7.

Vector calculus

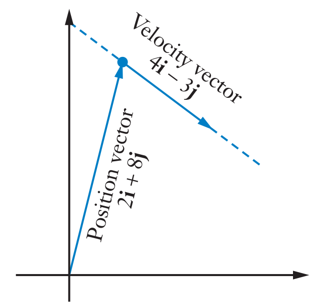
- Differentiating vector functions
- Integrating vector functions
- Some particular types of motion
- Miscellaneous exercise seven

Note

This chapter assumes that due to the probably concurrent study of Unit Three of the *Mathematics Methods* course, the reader is now familiar with the application of calculus to questions involving motion, and with differentiating and integrating trigonometrical functions and the exponential function, e^x.

In earlier chapters we encountered the idea of expressing the position vector of a moving object in terms of time, t.

For example, suppose that at some moment in time a ship is at a point with position vector $(2\mathbf{i} + 8\mathbf{j})$ km and is moving with velocity $(4\mathbf{i} - 3\mathbf{j})$ km/h (see diagram). If this motion continues, then t hours later the position vector of the ship will be



$$\begin{aligned}\mathbf{r} &= (2\mathbf{i} + 8\mathbf{j}) + t\,(4\mathbf{i} - 3\mathbf{j}) \\ &= (2 + 4t)\mathbf{i} + (8 - 3t)\mathbf{j}\end{aligned}$$

The position vector of the ship is given in terms of the scalar variable, t.
Each value of t will output one and only one $\mathbf{r}$.
We have a **vector function** given in terms of the scalar variable t.
Being a function of time we write the position vector as $\mathbf{r}(t)$.

Considering some general point P, cartesian coordinates (x, y), lying on the path of this ship

$$\begin{aligned}\mathbf{r} &= x\mathbf{i} + y\mathbf{j} \\ &= (2 + 4t)\mathbf{i} + (8 - 3t)\mathbf{j}\end{aligned}$$

Thus $\quad x = 2 + 4t$
and $\quad y = 8 - 3t \qquad \leftarrow$ The parametric equations of the path.

Eliminating t gives $\quad 4y = -3x + 38 \qquad \leftarrow$ The cartesian equation of the path.

Similarly, inclusion of a component in the $\mathbf{k}$ direction can define a path in three-dimensional space.

Given $\mathbf{r}(t)$, the position vector of an object as a function of time, we can consider differentiating this function to give the rate of change of $\mathbf{r}$ with respect to time.

Differentiating vector functions

WS Derivatives of vectors

If $\quad \mathbf{r} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$

the derivative $\dfrac{d\mathbf{r}}{dt}$ is determined by differentiating each component with respect to t.

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= \frac{d}{dt}(f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}) \\ &= \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}\end{aligned}$$

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For example, if $\qquad \mathbf{r} = 2t^3\mathbf{i} + (3t-1)\mathbf{j} - 2\mathbf{k}$

then $\qquad \dfrac{d\mathbf{r}}{dt} = 6t^2\mathbf{i} + 3\mathbf{j}$

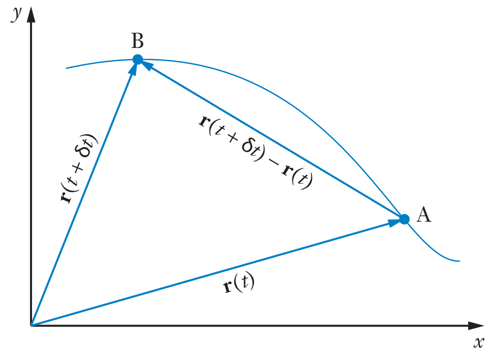
Note

- The derivative, $\dfrac{d\mathbf{r}}{dt}$, is the rate of change of $\mathbf{r}$ with respect to t.
- As t varies the position vector $\mathbf{r}(t)$ traces out a path. At any point on this path the vector $\dfrac{d\mathbf{r}}{dt}$ has the same direction as that of the tangent drawn at that point, as explained below.

$$\frac{d\mathbf{r}}{dt} = \lim_{\delta t \to 0} \frac{\mathbf{r}(t+\delta t) - \mathbf{r}(t)}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\overrightarrow{AB}}{\delta t} \qquad \text{see diagram.}$$

As $\delta t \to 0$ the direction of $\overrightarrow{AB}$ will tend towards that of the tangent at A.



- Whilst the position vector can be given in terms of any variable,

e.g $\qquad \mathbf{r} = 3u^2\mathbf{i} + 2u\mathbf{j} \qquad$ and thus $\qquad \dfrac{d\mathbf{r}}{du} = 6u\mathbf{i} + 2\mathbf{j},$

if $\mathbf{r}(t)$ is the position vector of an object at time t then $\dfrac{d\mathbf{r}}{dt}$ will be the velocity of the object at time t, and $\dfrac{d^2\mathbf{r}}{dt^2}$ its acceleration at time t.

EXAMPLE 1

A particle moves such that at time t seconds, $t \geq 0$, its position vector is $\mathbf{r}$ metres where

$$\mathbf{r} = 2\sin 3t\,\mathbf{i} + (2t^2 - 3t + 4)\mathbf{j}.$$

Find expressions for the velocity and acceleration of the particle at time t.

Solution

$$\mathbf{r} = 2\sin 3t\,\mathbf{i} + (2t^2 - 3t + 4)\mathbf{j}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

$$\therefore \quad \mathbf{v} = 6\cos 3t\,\mathbf{i} + (4t - 3)\mathbf{j}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

$$\therefore \quad \mathbf{a} = -18\sin 3t\,\mathbf{i} + 4\mathbf{j}$$

The particle has velocity $\mathbf{v}$ m/s and acceleration $\mathbf{a}$ m/s^2 at time t where

$$\mathbf{v} = 6\cos 3t\,\mathbf{i} + (4t-3)\mathbf{j} \qquad \text{and} \qquad \mathbf{a} = -18\sin 3t\,\mathbf{i} + 4\mathbf{j}.$$

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EXAMPLE 2

WS Position, velocity and acceleration

A particle moves such that at time t seconds, $t \geq 0$, its position vector is $\mathbf{r}$ m where

$$\mathbf{r} = 2t^2\mathbf{i} + (2t - 3)\mathbf{j}.$$

Find
- **a** the velocity of the particle when $t = 2$,
- **b** the speed of the particle when $t = 2$,
- **c** the angle that the velocity vector makes with the positive x direction when $t = 2$,
- **d** the acceleration of the particle when $t = 2$.

Solution

a

$$\begin{aligned} \mathbf{r} &= 2t^2\mathbf{i} + (2t - 3)\mathbf{j} \\ \mathbf{v} &= \frac{d\mathbf{r}}{dt} \\ &= 4t\mathbf{i} + 2\mathbf{j} \end{aligned}$$

When $t = 2$ $\quad \mathbf{v} = 8\mathbf{i} + 2\mathbf{j}$

The particle has velocity $(8\mathbf{i} + 2\mathbf{j})$ m/s when $t = 2$.

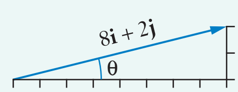
b When $t = 2$

$$\begin{aligned} |\mathbf{v}| &= \sqrt{8^2 + 2^2} \\ &= 2\sqrt{17} \end{aligned}$$

The speed of the particle when $t = 2$ is $2\sqrt{17}$ m/s.

c When $t = 2$ $\quad \mathbf{v} = 8\mathbf{i} + 2\mathbf{j}$

Thus if θ is the required angle, see diagram,



$$\begin{aligned} \tan\theta &= \frac{2}{8} \\ \theta &= 14.04°\text{, correct to two decimal places.} \end{aligned}$$

When $t = 2$ the velocity vector makes an angle of approximately 14° with the positive x direction. (Alternatively this answer could be obtained using the scalar product $\mathbf{v}.\mathbf{i}$.)

d

$$\begin{aligned} \mathbf{v} &= 4t\mathbf{i} + 2\mathbf{j} \\ \mathbf{a} &= \frac{d\mathbf{v}}{dt} \\ &= 4\mathbf{i} \end{aligned}$$

The particle has an acceleration of $4\mathbf{i}$ m/s^2 when $t = 2$.

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Integrals of vector functions

Integrating vector functions

To integrate a vector function we integrate each component.

If
$$\mathbf{r} = f(u)\mathbf{i} + g(u)\mathbf{j} + h(u)\mathbf{k}$$
$$\int \mathbf{r}(u)\,du = \left(\int f(u)\,du\right)\mathbf{i} + \left(\int g(u)\,du\right)\mathbf{j} + \left(\int h(u)\,du\right)\mathbf{k}$$

For example, if
$$\begin{aligned} \mathbf{r} &= 2u\mathbf{i} + (6u^2 + 1)\mathbf{j} \\ \int \mathbf{r}\,du &= (u^2 + c_1)\mathbf{i} + (2u^3 + u + c_2)\mathbf{j} \\ &= u^2\mathbf{i} + (2u^3 + u)\mathbf{j} + \mathbf{c} \qquad \text{where } \mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} \end{aligned}$$

In particular if $\mathbf{r}(t)$, $\mathbf{v}(t)$ and $\mathbf{a}(t)$ represent the position vector, velocity vector and acceleration vector at time t then

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt \quad \text{and} \quad \mathbf{v}(t) = \int \mathbf{a}(t)\,dt$$

EXAMPLE 3

A particle is initially at rest at a point with position vector $2\mathbf{j}$ m and t seconds later its acceleration is $\mathbf{a}$ m/s^2 where $\mathbf{a} = 8\cos 2t\mathbf{i} + 2\mathbf{j}$.

Find
- **a** an expression for the velocity of the particle at time t,
- **b** the value of t ($t > 0$) when the particle is first travelling parallel to the y-axis,
- **c** an expression for the position vector of the particle at time t.

Solution

a We know that
$$\begin{aligned} \mathbf{a} &= 8\cos 2t\mathbf{i} + 2\mathbf{j} \\ \therefore \quad \mathbf{v} &= \int (8\cos 2t\mathbf{i} + 2\mathbf{j})\,dt \\ &= 4\sin 2t\mathbf{i} + 2t\mathbf{j} + \mathbf{c} \end{aligned}$$
When $t = 0$, $\mathbf{v} = \mathbf{0}$. Thus
$$\begin{aligned} \mathbf{c} &= 0\mathbf{i} + 0\mathbf{j} \\ \therefore \quad \mathbf{v} &= 4\sin 2t\mathbf{i} + 2t\mathbf{j} \end{aligned}$$
The velocity of the particle at time t is $(4\sin 2t\mathbf{i} + 2t\mathbf{j})$ m/s.

b To be travelling parallel to the y-axis the $\mathbf{i}$ component of the velocity vector must be zero.

This occurs when
$$\begin{aligned} 4\sin 2t &= 0. \\ 2t &= 0, \pi, 2\pi, \ldots \\ t &= 0, \frac{\pi}{2}, \pi, \ldots \end{aligned}$$
For $t > 0$, the particle is travelling parallel to the y-axis, for the first time, when $t = \dfrac{\pi}{2}$.

c
$$\begin{aligned} \mathbf{r} &= \int (4\sin 2t\mathbf{i} + 2t\mathbf{j})\,dt \\ &= -2\cos 2t\mathbf{i} + t^2\mathbf{j} + \mathbf{d} \end{aligned}$$
When $t = 0$, $\mathbf{r} = 2\mathbf{j}$. Thus
$$2\mathbf{j} = -2\mathbf{i} + \mathbf{d}$$
Giving
$$\mathbf{d} = 2\mathbf{i} + 2\mathbf{j}$$
and so
$$\mathbf{r} = (2 - 2\cos 2t)\mathbf{i} + (t^2 + 2)\mathbf{j}$$
The position vector of the particle at time t is $(2 - 2\cos 2t)\mathbf{i} + (t^2 + 2)\mathbf{j}$ m.

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Exercise 7A

1 A particle moves such that at time t seconds, $t \geq 0$, its position vector is $\mathbf{r}$ metres where

$$\mathbf{r} = 2t^3\mathbf{i} + (3t + 1)\mathbf{j}.$$

a Find the initial position vector of the particle.

Use calculus to determine

b the velocity of the particle when $t = 3$,

c the speed of the particle when $t = 3$,

d the acceleration of the particle when $t = 3$.

2 A particle moves such that its acceleration, $\mathbf{a}$ m/s^2, at time t seconds, $t \geq 0$, is given by

$$\mathbf{a} = 6t\mathbf{i}.$$

Initially, i.e. when $t = 0$, the particle is at point A, position vector $(2\mathbf{i} - \mathbf{j})$ m, and is moving with velocity $(-4\mathbf{i} + 6\mathbf{j})$ m/s.

Find **a** the speed of the particle when $t = 2$,

b the distance the particle is from A when $t = 2$.

3 If $\mathbf{r} = 2t\mathbf{i} + (t - 1)\mathbf{j}$ find

a $\left|\dfrac{d\mathbf{r}}{dt}\right|$, **b** $\dfrac{d}{dt}|\mathbf{r}|$.

4 The vector functions $\mathbf{r}(t)$ m, $\mathbf{v}(t)$ m/s and $\mathbf{a}(t)$ m/s^2 are respectively the position, velocity and acceleration vectors of a particle at time t seconds.

Given that $\mathbf{v}(t) = \dfrac{-1}{(t+1)^2}\mathbf{i} + 2\mathbf{j}$ and $\mathbf{r}(0) = 3\mathbf{i} + 3\mathbf{j}$, determine

a $\mathbf{v}(1)$, **b** $\mathbf{a}(1)$, **c** $\mathbf{r}(1)$.

5 A particle moves such that at time t seconds, its position vector is $\mathbf{r}$ metres where

$$\mathbf{r} = (t^2 - 5t + 1)\mathbf{i} + (1 - 14t + t^2)\mathbf{j}.$$

a Find the value of t at the instant the particle is travelling parallel to the x-axis.

b Find the value of t at the instant the particle is travelling parallel to the y-axis.

6 The vector functions $\mathbf{r}(t)$ m, $\mathbf{v}(t)$ m/s and $\mathbf{a}(t)$ m/s^2 are respectively the position, velocity and acceleration vectors of a particle at time t seconds.

Given that $\mathbf{v}(t) = 2\mathbf{i} + e^{0.1t}\mathbf{j}$ and $\mathbf{r}(0) = 10\mathbf{j}$, determine

a $\mathbf{v}(10)$, **b** $\mathbf{a}(10)$, **c** $\mathbf{r}(10)$.

7 A particle moves such that at time t seconds its position vector is $\mathbf{r}$ m where

$$\mathbf{r} = (8t - 12)\mathbf{i} + t^2\mathbf{j}.$$

Find
- **a** how far the particle is from the origin when $t = 3$,
- **b** the velocity of the particle when $t = 3$,
- **c** the speed of the particle when $t = 3$,
- **d** the angle that the velocity vector makes with the positive x direction when $t = 3$. (Give your answer to the nearest degree.)

8 The vector functions $\mathbf{r}(t)$ m, $\mathbf{v}(t)$ m/s and $\mathbf{a}(t)$ m/s^2 are respectively the position, velocity and acceleration vectors of a particle at time t seconds $(t \geq 0)$.

Given that $\mathbf{r} = t^3\mathbf{i} + (2t^2 - 1)\mathbf{j}$ determine

- **a** the speed of the particle when $t = 2$,
- **b** the acceleration vector when $t = 3$,
- **c** the scalar product $\mathbf{v} . \mathbf{a}$ when $t = 2$,
- **d** the angle between $\mathbf{v}$ and $\mathbf{a}$ when $t = 2$ giving your answer in degrees correct to one decimal place.

9 A particle moves such that at time t seconds, $t \geq 0$, its velocity is $\mathbf{v}$ m/s where

$$\mathbf{v} = 2t\mathbf{i} + (3t^2 - 1)\mathbf{j} - 3\mathbf{k}.$$

Find
- **a** the initial speed of the particle,
- **b** the speed of the particle when $t = 2$,
- **c** the acceleration of the particle when $t = 2$,
- **d** the position vector of the particle when $t = 5$ given that when $t = 2$ the particle has position vector $(-4\mathbf{i} + 10\mathbf{j})$ m.

10 The position vector of a particle at time t seconds is $\mathbf{r}$ m where $\mathbf{r}$ is given by

$$\mathbf{r} = (t^2 - 6t - 16)\mathbf{i} + t^2\mathbf{j}.$$

For what value of t, $t \geq 0$, is

- **a** the particle on the y-axis?
- **b** the particle moving parallel to the y-axis?
- **c** the velocity of the particle perpendicular to the acceleration of the particle?

11 A position vector of a particle at time t seconds is $\mathbf{r}$ m where $\mathbf{r}$ is given by

$$\mathbf{r} = 3\mathbf{i} + 2t\mathbf{j} + (t^2 - 4t + 10)\mathbf{k}.$$

Find the position, velocity and acceleration of the particle at the instant that the particle is at its minimum distance from the x-y plane.

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12 With **i** and **j** horizontal and vertical unit vectors respectively, a body moves with a constant acceleration of $2\mathbf{j}$ m/s^2. Initially, when $t = 0$, the position vector of the body is $(\mathbf{i} + 20\mathbf{j})$ m and its velocity vector is $(2\mathbf{i} - 8\mathbf{j})$ m/s.

Find **a** the velocity of the body at time t seconds,

b the position vector of the body at time t seconds,

c the distance the body is from the origin when $t = 3$,

d the speed of the body when $t = 2$,

e the value of t when the body has its minimum height and determine this minimum height,

f the cartesian equation of the path of the body.

13 A particle moves with its acceleration, **a** m/s^2 at time t seconds given by

$$\mathbf{a} = \cos t\,\mathbf{i} + 2\mathbf{j}.$$

Initially, i.e. when $t = 0$, the particle has position vector $(4\mathbf{i} - 6\mathbf{j})$ m and velocity vector **j** m/s.

Find **a** the value(s) of t, $t \geq 0$, when the particle crosses the x-axis,

b the value(s) of t, $t \geq 0$, when the particle crosses the y-axis.

14 An object starts from rest at the origin and moves such that its acceleration t seconds later is **a** m/s^2, where **a** is given by

$$\mathbf{a} = -4\sin 2t\,\mathbf{i} + 2\mathbf{j} + e^t\mathbf{k}.$$

Find the position vector of the object when $t = \pi$.

15 A body moves such that its position vector, **r** m, at time t seconds is given by

$$\mathbf{r} = 2\sin 3t\,\mathbf{i} + 2\cos 3t\,\mathbf{j}.$$

a Find the value of t, $t \geq 0$, when the body crosses the x-axis for the first time.

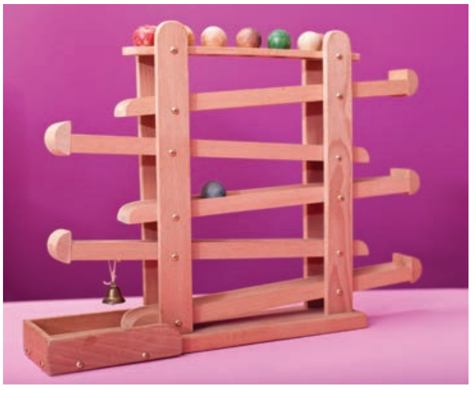
b Obtain expressions for the velocity and acceleration at time t.

c Prove that for all values of t the velocity is perpendicular to the acceleration.

16 An object is initially at a point A, position vector $(2\mathbf{i} + 8\mathbf{j})$ m, and moving with velocity $-4\mathbf{i}$ m/s. The object moves such that t seconds later its acceleration is **a** m/s^2, with

$$\mathbf{a} = 2\sin(0.5t)\mathbf{i} - 2\cos(0.5t)\mathbf{j}.$$

How far is the object from the point B, position vector $2\mathbf{i}$, when $t = \dfrac{\pi}{3}$?



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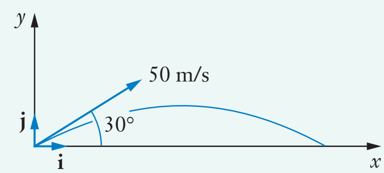
Some particular types of motion

EXAMPLE 4 **Particle projected from a point on a horizontal plane.**

A particle is projected from a point on a horizontal plane, at 30° above the horizontal and with an initial speed of 50 m/s. The particle will experience constant downward acceleration, due to the earth's gravitational pull, of g m/s^2.

With horizontal and vertical unit vectors **i** and **j** as shown, and taking g as 10, determine:



a the acceleration of the particle in **i**-**j** form,

b the initial velocity of the particle in **i**-**j** form,

c the velocity of the particle t seconds after projection,

d the time taken for the particle to reach its highest point.

Solution

a Acceleration of the particle $= -10\mathbf{j}$ m/s^2.

b Initial velocity of the particle $= (50\cos 30°\mathbf{i} + 50\sin 30°\mathbf{j})$ m/s
$= (25\sqrt{3}\mathbf{i} + 25\mathbf{j})$ m/s

c We know that $\mathbf{a} = -10\mathbf{j}$ where **a** m/s^2 is the acceleration.

Thus $\mathbf{v} = -10t\mathbf{j} + \mathbf{c}$ where **v** m/s is the velocity.

When $t = 0$, $\mathbf{v} = 25\sqrt{3}\mathbf{i} + 25\mathbf{j}$ $\quad\therefore\quad \mathbf{c} = 25\sqrt{3}\mathbf{i} + 25\mathbf{j}$

Thus $\mathbf{v} = 25\sqrt{3}\mathbf{i} + (25 - 10t)\mathbf{j}$

The velocity of the particle t seconds after projection is $[25\sqrt{3}\mathbf{i} + (25 - 10t)\mathbf{j}]$ m/s.

d At the highest point the vertical component of the velocity must be zero.

i.e. $25 - 10t = 0$
giving $t = 2.5$.

The particle is at its highest point 2.5 seconds after projection

Note

Readers with some knowledge of basic physics may recognise that part **d** of the above example could be solved using the formula, $v = u + at$. However remember that this formula is one of a group that apply to constant acceleration situations. The calculus techniques we are developing here apply to variable acceleration as well.

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Exercise 7B

Rectilinear motion with constant acceleration

1 At time $t = 0$ a particle is at the origin and moving with velocity $u\mathbf{i}$ m/s.

If the particle travels under the influence of a constant acceleration $a\mathbf{i}$ m/s^2 find expressions for the velocity and position vector of the particle t seconds later.

Particle projected from a point on a horizontal plane

2 An object is projected from a point O on a horizontal surface with an initial velocity of $(14\mathbf{i} + 35\mathbf{j})$ m/s where $\mathbf{i}$ and $\mathbf{j}$ are horizontal and vertical unit vectors respectively.

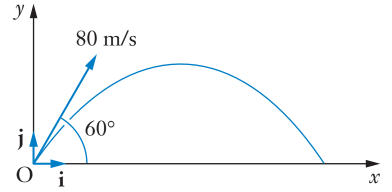
The object experiences constant downward acceleration of $-9.8\mathbf{j}$ m/s^2.

Find an expression for the position vector of the object, with respect to point O, t seconds into its flight.

How far is the object from O when $t = 5$?

Determine the cartesian equation of the path.
(Note: Your answer should be a quadratic, thus showing that the flight path is parabolic).

3 A particle is projected with an initial speed of 80 m/s at 60° above the horizontal from a point O on a horizontal surface. The particle experiences constant downward acceleration of g m/s^2.



With horizontal and vertical unit vectors $\mathbf{i}$ and $\mathbf{j}$ as shown, and taking g as 10, determine

a the acceleration of the particle in $\mathbf{i}$-$\mathbf{j}$ form,

b the initial velocity of the particle in $\mathbf{i}$-$\mathbf{j}$ form,

c the position vector of the particle, with respect to O, t seconds after projection,

d the time taken for the particle to return to the horizontal surface.

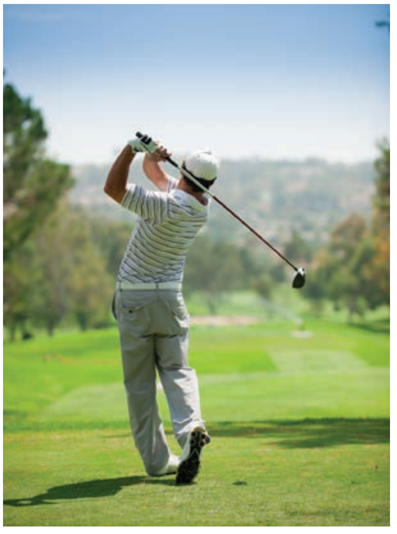
e the horizontal distance from projection to landing.

4 A golfer hits the ball from a point A towards some point B, on the same level as A and 120 m away. The ball has initial speed 42 m/s, at an angle θ above the horizontal. The horizontal and vertical unit vectors are $\mathbf{i}$ and $\mathbf{j}$ respectively and the ball experiences constant acceleration of $-9.8\mathbf{j}$ m/s^2.

a Find an expression for the position vector of the ball with respect to A, in terms of t, the time of flight, and θ.

b Determine the two possible values of θ that will cause the ball to land at B.

(Give your answers in degrees and rounded to one decimal place.)



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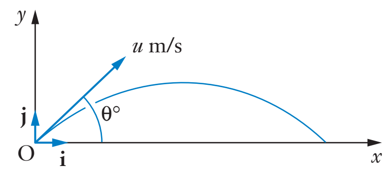
5 A particle is projected with an initial speed of u m/s at an angle $\theta°$ above the horizontal from a point O on a horizontal surface.

The particle experiences constant downward acceleration of g m/s^2.

With the unit vectors $\mathbf{i}$ and $\mathbf{j}$ as shown, determine the following (leaving u, θ and g in your answers)



a the velocity of the particle, in $\mathbf{i}$-$\mathbf{j}$ form, t seconds after projection,

b the position vector of the particle, in $\mathbf{i}$-$\mathbf{j}$ form, t seconds after projection,

c the time taken for the particle to return to the horizontal surface,

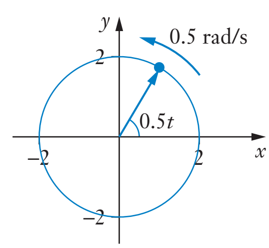
d the horizontal distance from O to the point of landing back on the horizontal surface,

e the value of θ that would make the distance of part **d** a maximum.

Circular motion with constant angular speed

6 Consider a particle initially at the point (2, 0) and moving around a circle of radius 2 m with constant angular speed 0.5 rad/s. The position vector of the particle, t seconds later, is $\mathbf{r}(t)$ m where

$$\mathbf{r}(t) = 2\cos(0.5t)\mathbf{i} + 2\sin(0.5t)\mathbf{j}.$$



a The velocity and acceleration vectors of the particle at time t seconds are $\mathbf{v}(t)$ m/s and $\mathbf{a}(t)$ m/s^2 respectively. Find $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

b Show that $|\mathbf{v}(t)|$ is independent of t and determine its value.

c Determine $\mathbf{v}\cdot\mathbf{a}$ and interpret the result.

d Show that $\mathbf{a}(t) = -k\mathbf{r}(t)$ for k a scalar constant and determine its value.

e What does the result $\mathbf{a}(t) = -k\mathbf{r}(t)$ mean in terms of the direction of $\mathbf{a}$?

7 (All units use metres and seconds.)

A particle moves in a circle, centre (0, 0).

The motion is such that $\mathbf{r}(0)$, the position vector when $t = 0$, is $5\mathbf{j}$.

The velocity at time t is given by

$$\mathbf{v}(t) = -\frac{5\pi}{2}\cos\left(\frac{\pi}{2}t\right)\mathbf{i} - \frac{5\pi}{2}\sin\left(\frac{\pi}{2}t\right)\mathbf{j}$$

a Determine $\mathbf{r}(t)$ the position vector of the particle at time t.

b Determine $\mathbf{r}(3)$.

c Sketch the path of the particle, indicating on your sketch the location and direction of motion of the particle at $t = 0$, $t = 1$ and at $t = 3$.

d Find $\int_0^3 \mathbf{v}(t)\,dt$, $\left|\int_0^3 \mathbf{v}(t)\,dt\right|$ and $\int_0^3 |\mathbf{v}(t)|\,dt$ and interpret each answer in terms of the motion of the particle from $t = 0$ to $t = 3$.

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Elliptical motion

8 An object moves such that its position vector, $\mathbf{r}(t)$ m, at time t seconds is given by

$$\mathbf{r}(t) = -2\sin\left(\frac{\pi}{6}t\right)\mathbf{i} + 3\cos\left(\frac{\pi}{6}t\right)\mathbf{j}.$$

a Produce a sketch of the motion for $0 \le t \le 12$ and indicate on your sketch the location and direction of motion of the object at $t = 0$, $t = 3$ and $t = 9$.

b With $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, determine the cartesian equation of the path of the object.

c Find the angle between the direction of the vector $\mathbf{i}$ and the direction of motion of the object when $t = 8$. (Answer in radians correct to 2 decimal places)

d If the acceleration of the object at time t seconds is $\mathbf{a}(t)$ m/s^2 show that

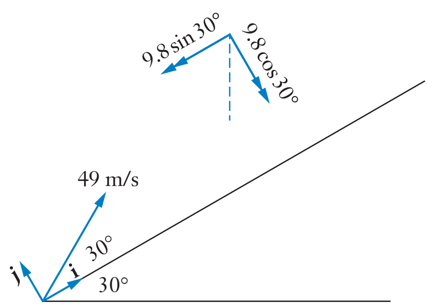
$$\mathbf{a}(t) = -k\mathbf{r}(t)$$

for k a positive scalar constant. Explain what this result means in terms of the direction of the acceleration and determine k.

Projectile up an inclined plane

9 A particle is projected directly up a plane that is inclined at 30° to the horizontal. The particle is projected with speed 49 m/s at 30° to the plane. If we take the unit vector $\mathbf{i}$ to be directly up the plane and the unit vector $\mathbf{j}$ to be perpendicular to the plane (see diagram) then the acceleration due to gravity will be

$$(-9.8\sin 30°\ \mathbf{i} - 9.8\cos 30°\ \mathbf{j})\ \text{m/s}^2.$$



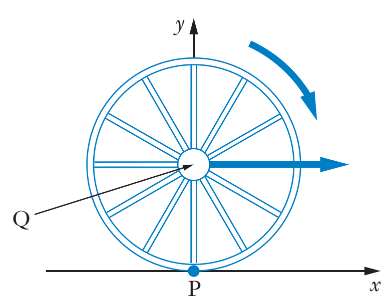
Taking the point of projection as (0, 0) obtain an expression for $\mathbf{r}(t)$, the position vector of the particle in the form $a\mathbf{i} + b\mathbf{j}$, t seconds after projection, and hence show that the particle will hit the plane $\dfrac{10\sqrt{3}}{3}$ seconds after projection.

Motion of a point on the rim of a rolling wheel

10 As the large wheel shown on the right rolls along the x-axis the point Q at the centre of the wheel will move horizontally. P is a point on the rim of the wheel and initially, i.e. when $t = 0$, point P lies at the origin. Suppose that the forward speed and the radius of the wheel are such that the velocity of P at time t seconds later is $\mathbf{v}$ m/s where

$$\mathbf{v} = (1 - \cos t)\ \mathbf{i} + \sin t\,\mathbf{j}$$

(Notice that writing this as $(\mathbf{i}) + (-\cos t\,\mathbf{i} + \sin t\,\mathbf{j})$ makes the separate translational and rotational components more obvious.)



a Determine the position vector of P at time t seconds.

b Find the diameter of the wheel.

c Find the position vector and the velocity of P when

i $t = 0$, **ii** $t = \dfrac{\pi}{2}$, **iii** $t = \pi$, **iv** $t = \dfrac{3\pi}{2}$.

d View the path of P on a graphic calculator.

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Miscellaneous exercise seven

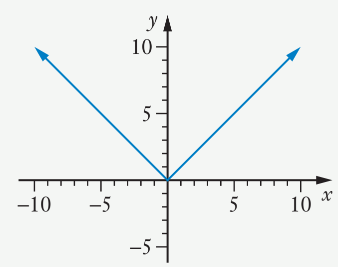
This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the Preliminary work section at the beginning of the book.

1 Express $(-\sqrt{3}+i)$ in the form $r\,\text{cis}\,\theta$ with $r \geq 0$ and $-\pi < \theta \leq \pi$.

2 Express the complex number $6\,\text{cis}\left(\frac{3\pi}{4}\right)$ in the form $a + bi$.

3 If $f(x) = \begin{cases} x \div 4 & \text{for} \quad x \leq 0 \\ x^2 & \text{for} \quad 0 < x < 3 \\ 3x & \text{for} \quad x \geq 3 \end{cases}$ determine $f^{-1}(x)$, the inverse of $f(x)$.

4 For which of the following functions is the graph as shown on the right?



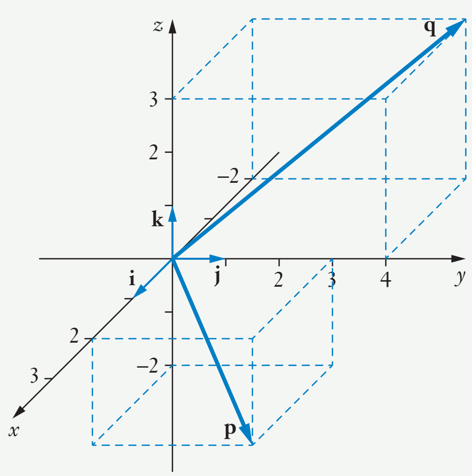
$f(x) = \sqrt{x^2}$

$g(x) = |x|$

$h(x) = \begin{cases} -x & \text{for} \quad x \leq 0 \\ x & \text{for} \quad x > 0 \end{cases}$.

5 If $f(x) = 3 + \sqrt{x+1}$ determine a formula for $f^{-1}(x)$, the inverse of $f(x)$, and state its domain and range.

6 Vectors **p** and **q** are as shown in the diagram on the right.



a Write **p** in the form $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

b Write **q** in the form $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

c Find, to the nearest degree, the angle between **p** and **q**.

d Find, to the nearest degree, the acute angle between **p** and the x-axis.

e Find, to the nearest degree, the acute angle between **q** and the y-axis.

7 Find a unit vector parallel to the resultant of $(\mathbf{i} + 5\mathbf{j} - 4\mathbf{k})$ and $(\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$.

8 Without the assistance of a calculator find $\mathbf{a} \times \mathbf{b}$ given $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$.

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9 State the domain and range of $g(f(x))$ if $f(x) = 5\sqrt{x}$ and $g(x) = \sqrt{4 - x}$.

10 Find the initial velocity, initial speed and initial acceleration of a particle given that its position vector t seconds into the motion is **r** metres where

$$\mathbf{r} = (6t + 1)\mathbf{i} + (t^3 + t^2 + 8t)\mathbf{j}.$$

11 On squared paper, and with an x-axis from -10 to 10
and a y-axis from 0 to 20,
accurately draw $y = |x - 5|$
and $y = |x + 5|$.
Hence draw $y = |x - 5| + |x + 5|$

For what values of x is $|x - 5| + |x + 5| \leq 14$?

12 If $z = a + ib$ and $\bar{z}$ is the complex conjugate of z find

a $z + \bar{z}$, **b** $z - \bar{z}$, **c** $z\bar{z}$, **d** $z \div \bar{z}$.

13 Point A has position vector $\begin{pmatrix} 1 \\ 6 \\ -7 \end{pmatrix}$ and point B has position vector $\begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$.

Find the position vector of the point P that divides AB internally in the ratio 4 : 1.

14 If $f(x) = x^2$ and $g(x) = \sqrt{x - 9}$ find the functions $f \circ g\,(x)$ and $g \circ f(x)$ in terms of x and state the natural domain and range of each.

15 If $f(x) = x^2$ and $g(x) = \sqrt{9 - x}$ find the functions $f \circ g\,(x)$ and $g \circ f(x)$ in terms of x and state the natural domain and range of each.

16 For each of the following conditions show diagrammatically the set of all points z lying in the complex plane and obeying the condition.

a $\operatorname{Re} z > \operatorname{Im} z$.

b $|z| \leq 3$.

c Both $3 \leq |z| \leq 5$ and $\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{4}$.

d $\arg(z - (2 + i)) = \frac{\pi}{4}$.

17 Show that the set of all points z in the complex plane that are such that

$$|z - 1| = 2|z - i|$$

together form a circle in the complex plane and find the centre and radius of the circle.

18 Find a unit vector perpendicular to $\begin{pmatrix} -8 \\ 4 \\ 1 \end{pmatrix}$.

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19 For $\{z: |z - 12 - 5i| = 4\}$ determine

a the minimum possible value of $\text{Im}(z)$.

b the maximum possible value of $|\text{Re}(z)|$.

c the maximum possible value of $|z|$.

d the minimum possible value of $|z|$.

e the minimum possible value of $\arg(z)$, giving your answer in radians correct to three decimal places.

f the maximum possible value of $\arg(z)$, giving your answer in radians correct to three decimal places.

20 Determine whether or not the lines L_1 and L_2 intersect and, if they do, determine the position vector of their point of intersection given that

L_1 has equation $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda(4\mathbf{i} + \mathbf{j} + 3\mathbf{k})$

and L_2 has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + \mu(3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$.

21 The graph on the right shows the curves

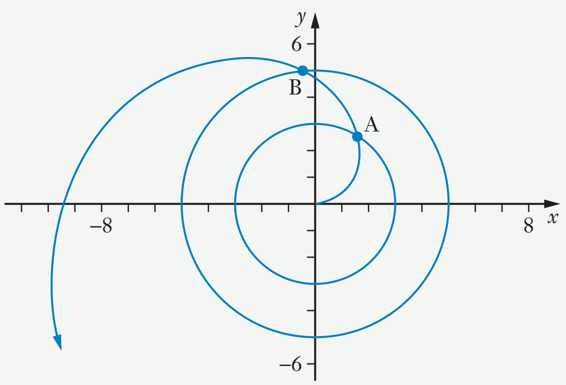
$|\mathbf{r}| = a$

$|\mathbf{r}| = b \qquad (b > a)$

and $|\mathbf{r}| = c\theta$

with a, b and c all positive integers.

Find a, b and c and the $(|\mathbf{r}|, \theta)$ coordinates (called polar coordinates) of points A and B, the points where two curves intersect.



22 Find the position vector of the point where the line L meets the plane Π given that L has equation

$\mathbf{r} = \begin{pmatrix} -1 \\ -10 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and Π has equation $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = 5$.

23 The four sets of points given below all represent the same straight line in the complex plane.

$\{z: |z - (3 + 5i)| = |z - (7 - i)|\}$ $\qquad$ $\{z: |z - (1 + 8i)| = |z - (a + bi)|\}$

$\{z: |z - 15| = |z - (c + di)|\}$ $\qquad$ $\{z: |z - 7 + 14i| = |z + e + fi|\}$

Find the values of a, b, c, d, e and f. (Hint: Use an Argand diagram.)

24 Express the vector $\begin{pmatrix} -6 \\ 5 \\ -3 \end{pmatrix}$ in the form $\lambda\mathbf{a} + \mu\mathbf{b} + \eta\mathbf{c}$ where

$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{c} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}.$

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25 Points A, B, C, D and E have position vectors

$$\begin{aligned}\mathbf{r}_A &= -\mathbf{i} + 6\mathbf{j} - 8\mathbf{k},\\ \mathbf{r}_B &= 3\mathbf{i} + 4\mathbf{k},\\ \mathbf{r}_C &= 7\mathbf{i} + c\mathbf{k},\\ \mathbf{r}_D &= -\mathbf{i} + d\mathbf{j} - 5\mathbf{k},\\ \mathbf{r}_E &= 4\mathbf{i} + \mathbf{j} + e\mathbf{k}.\end{aligned}$$

Find the position vector of point F, the midpoint of AB.

A sphere has its centre at point F and all of the five points A, B, C, D and E lie on the surface of the sphere. Determine the values of c, d and e given that they are all non-negative constants.

26 **a** Find, in radians and correct to two decimal places, the acute angle between the lines L_1 and L_2 given that

L_1 has equation $\mathbf{r} = -2\mathbf{i} + 3\mathbf{j} + 8\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{k})$

and L_2 has equation $\mathbf{r} = 5\mathbf{i} + 6\mathbf{j} - 3\mathbf{k} + \mu(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$.

b Prove that L_1 and L_2 intersect and find the position vector of point P, the point of intersection.

c Find, in scalar product form, the vector equation of the plane containing point P and perpendicular to the line joining point A, position vector $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, to point B, position vector $3\mathbf{i} + \mathbf{j} + \mathbf{k}$.

27 Use the idea of proof by contradiction to prove that the lines

L_1: $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$

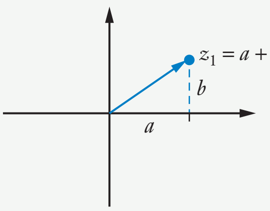
and L_2: $\mathbf{r} = 8\mathbf{i} + \mu(2\mathbf{i} + \mathbf{j} - 3\mathbf{k})$

do not intersect.

28 Express $\cos 4\theta$ in terms of $\cos\theta$.

29 z_1 shown in the diagram on the right is one solution to the equation $z^4 = k$.

Find z_2, z_3 and z_4, the other three solutions to the equation, giving all answers in cartesian form in terms of a and b.



30 Solve each of the following systems of equations without the assistance of a calculator. Your method should clearly indicate the steps you take to eliminate variables.

a
$$\begin{aligned} x + 2y + z &= 7\\ x + 3y + 2z &= 11\\ 2x + 5y + 5z &= 26\end{aligned}$$

b
$$\begin{aligned} 2x + 3y + 5z &= 4\\ x + 2z &= 1\\ 3x + y + 7z &= 3\end{aligned}$$

31 For $0 < t < \pi$, an object moves such that its position vector, $\mathbf{r}(t)$, is given by

$$\mathbf{r} = 2\sec\left(t - \frac{\pi}{2}\right)\mathbf{i} + 4\tan\left(t - \frac{\pi}{2}\right)\mathbf{j}.$$

With $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, find the Cartesian equation of the path of the object for $0 < t < \pi$.

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32 A glider is following a straight line flight path to touch down at $(0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k})$ m. The vector equation of this straight line is $\mathbf{r} = \lambda(10\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$ and the ground is the **i**-**j** plane.

Initially the glider has an altitude of 180 m (with respect to the touchdown point) and it touches down 15 seconds later.

Find the velocity of the glider during the 15 seconds (assume this velocity is constant) and the distance the glider travels in this time (to the nearest ten metres).

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33 Prove that the only requirement necessary for the system of equations shown below to have a unique solution is that p must not equal 6.

$$\begin{aligned} x + y + (p-3)z &= 1 \\ 2x + 4y + 4z &= 3 \\ -x + y + (7-2p)z &= q \end{aligned}$$

a If p does equal 6, find the value(s) of q for the system to have

i infinite solutions, **ii** no solutions.

b If p = 5 find the unique solution in terms of q.

c If p = 6 and q takes the value that gives infinite solutions, find the particular solution for which $x = 1$.

34 An object moves such that its position vector **r** m, at time t s, is such that the velocity vector, $\dot{\mathbf{r}}$ m/s, is given by

$$\dot{\mathbf{r}} = 4\cos 2t\mathbf{i} + 3\mathbf{j} \qquad (t \geq 0).$$

a When $t = 0$ the object has position vector $(2\mathbf{i} - \mathbf{j})$ m, with respect to an origin, O. Find the position vector of the object when $t = \pi$.

b Find the speed and position vector of the object at the first time, $t > 0$, for which the velocity of the object is perpendicular to the acceleration of the object.

35 Prove that the following system of three equations in three unknowns

$$\begin{aligned} x + 2y + z &= 3 \\ -x + (p-2)y + (q-1)z &= 0 \\ x + (r+2)y + (s+1)z &= 5 \end{aligned}$$

must have ps ≠ qr for there to be a unique solution.

36 Find the shortest distance from the line

$$\mathbf{r} = \begin{pmatrix} -3 \\ -7 \\ 8 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$$

to the point with position vector $\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$.

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37 An object is projected from a point (0, 0), on horizontal ground, with an initial speed of u m/s at an angle θ above the horizontal. Use as your starting point the fact that this object will experience a constant acceleration of $-g\mathbf{j}$ m/s^2 and then use calculus to determine

a an expression for the position vector of the object t seconds after projection,

b the possible values of θ if an object projected with speed 50 m/s is to pass through a point with position vector $(100\mathbf{i} + 40\mathbf{j})$ metres. Use $g = 10$ and give your answers in degrees correct to one decimal place.

38 Given the system of equations:

$$\begin{aligned} x + 2y + z &= 2 \\ x - 3y + 2z &= 10 \\ 2x + 3y + z &= 2 \\ x + 5y + 4z &= 8 \end{aligned}$$

a student writes the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 1 & -3 & 2 & 10 \\ 2 & 3 & 1 & 2 \\ 1 & 5 & 4 & 8 \end{array}\right]$$

and correctly reduces it to:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -5 & 1 & 8 \\ 0 & 0 & 6 & 18 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus reducing the initial four equations to:

$$\begin{aligned} x + 2y + z &= 2 \\ 0x - 5y + z &= 8 \\ 0x + 0y + 6z &= 18 \\ 0x + 0y + 0z &= 0 \end{aligned}$$

Noticing the last of these equations the student concludes that the initial system has no solutions. Was the student correct in this conclusion? Explain your answer and, if you disagree with the student's conclusion, state clearly what you think the conclusion should be.

39 A golfer hits a ball from point T, giving it an initial velocity $(30\mathbf{i} + 24\mathbf{j})$ m/s where $\mathbf{i}$ and $\mathbf{j}$ are horizontal and vertical unit vectors respectively.

The ball lands at a point L where L has a position vector relative to T of $(135\mathbf{i} + c\mathbf{j})$ m.

Throughout its flight the ball is subject to an acceleration of $-10\mathbf{j}$ m/s^2, due to gravity.

a Find an expression for the velocity of the ball t seconds into its flight.

b Find an expression for the position vector of the ball, relative to point T, t seconds into its flight.

c What time passes from the ball leaving T to it reaching the highest point in its path?

d What time passes from the ball leaving T to it reaching L?

e What is the greatest height reached by the ball?

f Find the value of c.

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40 An earlier chapter asked you to use proof by induction to prove de Moivre's theorem

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta),$$

for positive integer values of n.

Now suppose that n is a negative integer, i.e. $n = -k$ for k a positive integer.

Prove that

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta),$$

for n a negative integer.

41 With the complex numbers

$$z_1 = \sqrt{6}\,\text{cis}\left(\frac{5\pi}{6}\right) \qquad z_2 = 2\,\text{cis}\left(\frac{\pi}{2}\right) \qquad \text{and} \qquad z_3 = 3\,\text{cis}\left(\frac{2\pi}{3}\right),$$

and without the assistance of a calculator, simplify $\left(\dfrac{z_1}{z_2 z_3}\right)^{-3}$.

42 Determine, with justification, whether the system of equations shown below has a unique solution, no solution or infinite solutions and, if there is a unique solution, determine that solution.

$$\begin{aligned} x + 3y - z &= 3 \\ -x - 3y + z &= 3 \\ 2x + 6y - 2z &= 6 \end{aligned}$$

43 For the system of equations:

$$\begin{aligned} 3x + 2y + z &= 4 \\ x - y + 2z &= 3 \\ 3x + 7y + \text{p}z &= \text{q} \end{aligned}$$

a determine the value(s) of p and q for there to be infinite solutions,

b determine the value(s) of p and q for there to be no solutions.

c If $\text{p} = k$ and $\text{q} = k + 7$ and the system has a unique solution of $\begin{cases} x = m \\ y = n \\ z = 3 \end{cases}$, find m, n, p and q.

44 If we take the origin as the point where the post supporting my mailbox meets the ground, then my movement-activated light is situated at a point with position vector $\begin{pmatrix} -1 \\ 8 \\ 5 \end{pmatrix}$ m.

The light is set to switch on if a person, or sufficiently large animal, comes within 6 metres of the light.

A 'sufficiently large' dog walks up the sloping verge at the front of the house and follows the line with vector equation

$$\mathbf{r} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

If the dog continues along this line does it cause the light to switch on? (Justify your answer.)

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